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On the Equivalence of the Maxwell-Chern-Simons Theory and a Self-Dual Model

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Abstract

We study the connection between the Green functions of the Maxwell-Chern-Simons theory and a self-dual model by starting from the phase-space path integral representation of the Deser-Jackiw master Lagrangian. Their equivalence is established modulo time-ordering ambiguities.

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In a recent interesting paper [1] the bosonization of the massive Thirring model in 2+1 dimensions was discussed by relating it in the large mass limit to the Maxwell-Chern-Simons (MCS) theory [2]. As an intermediary step use has been made of the equivalence [3] of this theory to that of a self-dual (SD) model discussed in [4]. This analysis has been carried out on the level of the configuration space path-integral expressions of the partition functions. Because of the constraint structure associated with the various Lagrangians involved in the argument, a complete investigation of the problem must start from a proper phase-space path-integral formulation. This is done in the present note. Starting from the master Lagrangian of Deser and Jackiw [3] we follow the general line of reasoning of ref. [1] and establish the equivalence, modulo time-ordering ambiguities, of the SD model and the MCS theory on the level of Green functions.

Consider the symmetrized form of the master Lagrangian given in [3]

$$\mathcal{L} = \frac{1}{2} f^\mu f_\mu - \frac{1}{2}\epsilon^{\mu\nu\lambda} f_\mu \partial_\nu A_\lambda - \frac{1}{2}\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu f_\lambda + \frac{m}{2}\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \tag{1}$$

The primary constraints [5] are given by

$$\begin{aligned} \Omega_0 &= \pi_0 \approx 0; \quad \Omega_i = \pi_i + \frac{1}{2}\epsilon_{ij} f^j - \frac{m}{2}\epsilon_{ij} A^j \approx 0 \\ \Omega_0^{(f)} &= \pi_0^{(f)} \approx 0; \quad \Omega_i^{(f)} = \pi_i^{(f)} + \frac{1}{2}\epsilon_{ij} A^j \approx 0 \end{aligned} \tag{2}$$

where $\pi_\mu(\pi_\mu^{(f)})$ are the momenta canonically conjugate to $A^\mu(f^\mu)$. The canonical Hamiltonian is given by

$$H_c = \int d^2x \left[-\frac{1}{2} f^\mu f_\mu + A_0 \epsilon^{ij} \left(\partial_i f_j - m \partial_i A_j \right) + \epsilon^{ij} f_0 \partial_i A_j \right] \tag{3}$$

Persistency of the first-class constraints Ω_0 and $\Omega_0^{(f)}$ in time leads, respectively, to the following secondary constraints

$$\begin{aligned} \Omega_3 &= \epsilon^{ij}\partial_i f_j - m\epsilon^{ij}\partial_i A_j \approx 0 \\ \Omega_3^{(f)} &= f_0 - \epsilon^{ij}\partial_i A_j \approx 0 \end{aligned} \tag{4}$$

Although, apart from π_0, all other constraints appear to be second class, there actually exists a linear combination of the constraints which is first class. This constraint is given by

$$\Omega = \vec{\nabla} \cdot \vec{\Omega} + \Omega_3 \approx 0, \tag{5}$$

which can be checked to be the generator of the gauge transformations $A^i \to A^i + \partial^i \lambda$, $f^i \to f^i$. There are no further constraints. Hence we have two first-class constraints, Ω_0 and Ω, and six second-class constraints $\Omega_0^{(f)}, \Omega_i^{(f)}, \Omega_3^{(f)}$

and Ω_i. Since the equivalence to be demonstrated refers to the observables of the SD and MCS models, we are free to choose the Coulomb gauge for our discussion. The phase-space partition function [6] in this gauge is then given by

$$
\begin{aligned}
Z &= \int Df^\mu D\pi_\mu^{(f)} DA^\mu D\pi_\mu \delta(A_0)\delta(\vec{\nabla}\cdot\vec{A})\delta(\Omega_0)\delta(\Omega)\delta(\Omega_i) \\
&\quad \prod_{\alpha=0}^{3}\delta(\Omega_\alpha^{(f)})e^{\int d^3x\left[\pi_\mu \dot{A}^\mu + \pi_\mu^{(f)}\dot{f}^\mu - \mathcal{H}_c\right]}
\end{aligned} \tag{6}
$$

The Faddeev-Popov determinants associated with the constraints and the gauge-fixing are all trivial, and hence do not appear in the functional integral. The momentum integrations in (6) can be easily performed and one obtains

$$
Z = \int df^\mu DA^\mu \delta(\Omega_3^{(f)})\delta(\vec{\nabla}\cdot\vec{A})e^{i\int d^3x\mathcal{L}_M} \tag{7}
$$

To arrive at (7) we have expressed $\delta(\Omega_3)$ as a Fourier integral and have redefined the A^0 field in order to obtain a manifestly Lorentz-covariant action. We next couple the gauge-invariant fields f^μ and $F^\mu = \epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda$ to external sources in order to establish the equivalence of the MCS and SD models on the level of Green functions. From (7) we are led to consider the generating functional

$$
Z[J,j] = \int Df^\mu DA^\mu \delta(\Omega_3^{(f)})\delta(\vec{\nabla}\cdot\vec{A})e^{i\int d^3x[\mathcal{L}+J_\mu F^\mu + j_\mu f^\mu]} \tag{8}
$$

The f_μ-integration is easily done to yield

$$
Z[J,j] = \int DA^\mu \delta(\vec{\nabla}\cdot\vec{A})e^{i\int d^3x[\mathcal{L}_{MCS}+F_\mu(J^\mu+j^\mu)+\frac{1}{2}\vec{j}^2]} \tag{9}
$$

where

$$
\mathcal{L}_{MCS} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \tag{10}
$$

is the familiar MCS Lagrangian. For vanishing sources this is the partition function of the MCS theory in the Coulomb gauge.

Alternatively, one may perform the A_μ integration. To this end we first integrate over the A_0 field, leading to

$$
\begin{aligned}
Z[J,j] &= \int \mathcal{D}f^\mu \mathcal{D}A^i \delta(\vec{\nabla}\cdot\vec{A})\delta(\Omega_3^{(f)})\delta(mf_0 - \epsilon^{ij}\partial_i f_j + \epsilon_{ij}\partial^i J^j) \\
&\quad \times e^{i\int d^3x[\mathcal{L}' + f_\mu j^\mu + F_0 J^0 - \epsilon_{ij}J^i\partial_0 A^j]}
\end{aligned} \tag{11}
$$

where

$$
\mathcal{L}' = \frac{1}{2}f_\mu f^\mu - \frac{m}{2}\epsilon^{ij}A_i\partial_0 A_j - \epsilon^{ij}(f_0\partial_i A_j - f_i\partial_0 A_j) \tag{12}
$$

The Gaussian A^i integration may be performed by expanding the A^i fields about the classical solution of the constraint equation in the Coulomb gauge,

$$A_i^{cl}(\vec{x},t) = \epsilon_{ij}\partial^j \int d^2\vec{x}' D(\vec{x}-\vec{x'}) f_0(\vec{x'},t) \tag{13}$$

where $\vec{\nabla}^2(D(x-\vec{x'}) = \delta(\vec{x}-\vec{x'})$. One then finds that

$$\begin{aligned} Z &= \int \mathcal{D}f_\mu \delta(mf_0 - \epsilon^{ij}\partial_i f_j + \epsilon_{ij}\partial_i J^j) \\ &\quad \times \exp\left\{ i\int \mathcal{L}_{SD} + f_\mu(J^\mu + \frac{1}{m}\epsilon^{\mu\nu\lambda}\partial_\nu J_\lambda\right) - \frac{1}{2m}\epsilon^{\mu\nu\lambda} J_\mu \partial_\nu j_\lambda \end{aligned} \tag{14}$$

where

$$\mathcal{L}_{SD} = \frac{1}{2} f_\mu f^\mu - \frac{1}{2m}\epsilon^{\mu\lambda\nu} f_\mu \partial_\lambda f_\nu \tag{15}$$

is the self-dual Lagrangian of [4]. We note that the source J^i appears in the argument of the delta-function. A more convenient form for the computation of Green functions is obtained by performing the integration over f_0. Then it can be verified that the resulting path integral expression can also be written in the form

$$\begin{aligned} Z[j,J] &= \int \mathcal{D}f_\mu \delta(mf_0 - \epsilon^{ij}\partial_i f_j) e^{i\int \mathcal{L}_{SD}} \\ &\times \ \exp\left\{ i\int \tilde{f}_\mu J^\mu + j^\mu f_\mu - \frac{1}{2m}\epsilon^{\mu\nu\lambda} J_\mu \partial_\nu J_\lambda - \frac{1}{2m^2}(\epsilon^{ij}\partial_i J_j)^2 \right. \\ &\quad \left. -\frac{1}{m} j^0 \epsilon^{ij}\partial_i J_j \right\} \end{aligned} \tag{16}$$

where

$$\tilde{f}_\mu = \frac{1}{m}\epsilon_{\mu\nu\lambda}\partial^\nu f^\lambda \tag{17}$$

is the dual of f_μ.

In the absence of sources, expressions (14) or (16) reduce to the partition function associated with the SD-Lagrangian. Recalling the alternative representation (9), we infer from here the equivalence of the partition functions corresponding to the MCS and SD models.

We next consider this equivalence on the level of Green functions. Because of the Gaussian character of the models, it is sufficient to consider the respective two-point functions. Functionally differentiating the partition functions (9) and (16) with respect to the sources j^μ and j^ν, we obtain

$$< F_\mu(x) F_\nu(y) >_{MCS} - i\delta_{\mu i}\delta_{\nu i}\delta(x-y) = < f_\mu(x) f_\nu(y) >_{SD} \tag{18}$$

Alternatively, by functionally differentiating (9) and (16) with respect to the sources J^μ and J^ν one finds that

$$< F_\mu(x) F_\nu(y) >_{MCS} - i\delta_{\mu i}\delta_{\nu i}\delta(x-y) = < \tilde{f}_\mu(x) \tilde{f}_\nu(y) >_{SD} + S_{\mu\nu}(x-y) \tag{19}$$

where

$$S_{\mu\nu}(x-y) = -\frac{i}{m}\epsilon_{\mu\nu\lambda}\partial^\lambda\delta(x-y) - \frac{i}{m^2}\epsilon_{0\mu\lambda}\epsilon_{0\nu\rho}\partial^\lambda\partial^\rho\delta(x-y) - i\delta_{\mu i}\delta_{\nu i}\delta(x-y) \tag{20}$$

Finally, by differentiating (9) and (16) with respect to j^μ and J^ν one is led to the relation

$$< F_\mu(x)F_\nu(y) > -i\delta_{\mu i}\delta_{\nu i}\delta(x-y) =< f_\mu(x)\tilde{f}_\nu(y) > +S'_{\mu\nu}(x-y) \tag{21}$$

where

$$S'_{\mu\nu}(x-y) = -i\delta_{\mu i}\delta_{\nu i}\delta(x-y) + \frac{i}{m}\delta_{\mu 0}\delta_{\nu j}\epsilon_{jk}\partial_k\delta(x-y) \tag{22}$$

Note that the contact term, proportional to the δ-function, contributes in the same way in all three relations. Moreover, the remaining Schwinger-like terms appearing in (20) and (22) can be recognized as arising from a time-ordering ambiguity in the $\tilde{f}_\mu$ fields. This can be verified by expressing these fields in terms of the f_μ's, which are actually coupled to the sources j^μ, and making use of the commutators of the f_μ fields given in [3]. From eqs. (18-22)we conclude that modulo a contact term and time-ordering ambiguities the following identifications hold

$$F^\mu \leftrightarrow f^\mu \leftrightarrow \tilde{f}^\mu$$

On the classical level this correspondence follows from the equations of motion derived from the master Lagrangian (1) [3]. The present note therefore rounds up the analysis of refs. [3] and [1].

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